Recall that $S(X_{ét})$ has enough injectives, so we can work with right derived functors.

Examples

1) Consider the functor $\Gamma(X,-) : S(X_{et}) \longrightarrow Ab$. Then $R^i\Gamma(X,F) = H^i(X,F)$ the cohomology. Given a s.e.s, we get δ-functors for a l.e.s.
2) Consider $f: Y \to X$ of schemes. Then $f_*: S(Y_{et}) \longrightarrow S(X_{et})$ by $(f_*F)(U) = F(U \times_X Y)$. This has a left adjoint f^*, and also have higher direct images R^if_*.
3) We have morphisms $X_{fl} \xrightarrow{\pi} X_{et} \xrightarrow{\pi_0} X_{Zar}$, which can also be derived. We say a flabby sheaf is if $\forall U \to X$, $H^i(U_{et}, F|_U) = 0$, $i > 0$. Then π_* preserves flabby sheaves.

Thm:

1) For any map of sites $\pi : X'_{E'} \to X_E$, we have the Leray spectral sequence with $E_2^{p,q} = H^p(X_E, R^q\pi_* F) \Rightarrow H^{p+q}(X'_{E'}; F)$.
2) For any triple $X''_{E''} \xrightarrow{\pi'} X'_{E'} \xrightarrow{\pi} X_E$, we have the Grothendieck spectral sequence with $E_2^{p,q} = (R^p\pi_*)(R^q\pi'_*)(F) \Rightarrow R^{p+q}(\pi\pi')_* F$.

Proof: Thm 1.18 in chap 3, EC. ☒

These can be used to give a comparison Theorem:

Thm: $H^i(X_{Zar}, F) = H^i(X_{fl.}, W(F))$, Where $W(F)(U \xrightarrow{fl} X) = \Gamma(U, f^*F)$.

Proof: Since $H^p(X_{Zar}, R^q\pi_* W(F)) \Rightarrow H^{p+q}(X_{fl}, W(F))$, its enough to show $R^q\pi_* W(F) = 0$ for $q > 0$. One can compute this w/ Čech cohomology. See EC. ■

Thm: Let G be a quasi-proj X-scheme, which is a group scheme. Then calling by G the sheaf G represents, $H^i(X_{et}, G) = H^i(X_{fl}, G)$.

Etale & Complex Cohomology

Let X be a smooth scheme over Spec $\mathbb{C}$. Then for any finite group Λ, we have $H^i_{sing}(X(\mathbb{C}), \Lambda) \cong H^i(X_{et}, \Lambda)$.

We usually take $\Lambda = \mathbb{Z}/\ell\mathbb{Z}$, ℓ prime, and the inverse system with surjective maps:

$$\cdots\cdot \mathbb{Z}/\ell^2\mathbb{Z} \to \mathbb{Z}/\ell\mathbb{Z}.$$

Then we can check whether or not $H^i(-, \varprojlim \mathbb{Z}/\ell^n\mathbb{Z}) \cong \varprojlim H^i(-, \mathbb{Z}/\ell^n\mathbb{Z})$. In the surjective case, we are good.

Def: $H^i(X_{et}, \mathbb{Z}_\ell) = \varprojlim H^i(X_{et}, \mathbb{Z}/\ell^n\mathbb{Z})$, and $\mathbb{Q}_\ell$-étale cohomology is given by: $H^i(X_{et}, \mathbb{Q}_\ell) = H^i(X_{et}, \mathbb{Z}_\ell) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$.

Now let G be a profinite group (cf. Gal (k_s/k)) and consider discrete G-modules M, and this category is abelian with enough injectives. We know $M \mapsto M^G$ is left exact, and the derived functors give the Galois cohomology $H^i(G; M)$.

Prop: $H^q(G; M) = \varinjlim_{\substack{U \text{ open, normal} \\ \text{subgrp}}} H^q(G/U; M)$.

Corollary: $H^q(G; M)$ is a torsion group.

One can compute using the complex of continuous cochains. This is the analogue of Čech cohomology. See Serre, "Galois Cohomology".

Now consider a group $U \triangleleft G$, M a G-module. Decompose $M \mapsto M^G$ by $M \to M^U \mapsto (M^U)^{G/U}$, which is a composition of two left exact functors. The spectral sequence is the Hochschild-Serre spectral sequence.

If $X = \operatorname{Spec} k$, we know $S(X_{et}) \simeq G_k\text{-mod}$. Then $H^i(X_{et}, F) = H^i(G_k, M_F)$, where $M_F = F_{\bar{x}}$, where $\bar{x} \hookrightarrow \operatorname{Spec} k$ is a geometric point.

Prop: Let $X' \xrightarrow{\pi} X$ be a finite Galois covering with Galois grp G. Then there is a spectral sequence (Hochschild-Serre) with $H^p(G, H^q(X'_{et}, F)) \Rightarrow H^{p+q}(X_{et}, F)$.

Example

Let $X' \xrightarrow{f} X$ be an infinite Galois covering (inverse limit of finite galois coverings) with Galois group G. Then we want $H^{p+q}(X_{et}, F) \Leftarrow H^p(G, H^q(X', f^*F))$. This indeed exists, via a limit of spectral sequences.

A Computation

Let X be a scheme. We want to compute $H^q(X_{et}, \mathbb{G}_m)$. Some setup: X is an integral, regular, quasi-compact scheme. Then we have a generic point $g : \eta \hookrightarrow X$, $\eta = \operatorname{Spec} K$, K is the field of rational functions on X. We get a sequence

$$1 \longrightarrow \mathbb{G}_{m,X} \longrightarrow g_* \mathbb{G}_{m,\eta}$$

with cokernel $\overset{\text{"}D_X\text{"}}{\overset{\parallel}{\bigoplus i_{v*} \mathbb{Z}}}$, with v running over all codim $= 1$ points of X, $i_v : v \hookrightarrow X$.

Step 1: Compute $H^i(X, D_X) = \bigoplus_{v \in X_1} H^i(X, i_{v*}\mathbb{Z})$. Note $H^0(X, i_{v*}\mathbb{Z}) = H^0(v, \mathbb{Z}) = \mathbb{Z}$, almost by definition. Now we look at the Leray S.S. $f : Y \to X$, $F \in S(Y_{et})$. Then:

$$\begin{tikzcd}
E_2 : H^0(X, R^2 f_* F) \arrow[rrd, "d_2"] & & \\
H^0(X, R^1 f_* F) \arrow[rrd, "d_2"] & H^1(X, R^1 f_* F) & H^2(X, R^1 f_* F) \\
H^0(X, f_* F) & H^1(X, f_* F) & H^2(X, f_* F)
\end{tikzcd}$$

Get $0 \to H^1(X, f_* F)$
$\to H^1(Y, F) \to H^0(X, R^1 f_* F)$
$\to H^2(X, f_* F) \to H^2(Y, F)$.

Now apply this to $f = i_v : \operatorname{Spec} k(v) \longrightarrow X$, $F = \mathbb{Z}$. We get:

$$(*) \quad 0 \to H^1(X, i_{v*}\mathbb{Z}) \longrightarrow H^1(v, \mathbb{Z}) \longrightarrow H^0(X, R^1 i_{v*}\mathbb{Z}) \longrightarrow H^2(X, i_{v*}\mathbb{Z}) \longrightarrow H^2(v, \mathbb{Z}).$$

We claim $R^1 i_{v*} \mathbb{Z} = 0$.

Lemma: Let $f : Y \to X$ be a morphism and $F \in S(Y_{et})$. Then for any $i \geq 0$, $R^i f_* F$ is the sheafification of the presheaf $(U \xrightarrow{et} X) \longmapsto H^i(U \times_X Y, F)$.

Proof: Choose an inj. resolution $0 \to F \to I^\bullet$. Then $R^i f_* F =$ ith cohomology of $f_* I^\bullet$, which is the sheaf associated to the claimed presheaf. ■

So to prove the claim, we show a vanishing result. Let $U \xrightarrow{et} X$, and it suffices to show $H^i(U \times \nu, \mathbb{Z}) = 0$. So:

$$\begin{tikzcd} \underset{\substack{k/k(\nu) \\ \text{fin. sep.}}}{\coprod} \operatorname{Spec} k = U \times_X \nu \arrow[r, "et"] \arrow[d] & \nu = \operatorname{Spec} k(\nu) \arrow[d, hook] \\ U \arrow[r, "et"] & X \end{tikzcd}$$

$$\Bigg\} \Rightarrow \bigoplus_{k} H^1(\operatorname{Spec} k, \mathbb{Z}) = H^1(U \times \nu, \mathbb{Z}).$$
$$= H^1(G_k, \mathbb{Z}).$$

Exercise 24: Let M be a trivial G_k-module. Then $H^1(G_k, M) = \operatorname{Hom}_{cont}(G_k, M)$

$\Rightarrow H^1(G_k, \mathbb{Z}) = 0$. Now $H^1(\nu, \mathbb{Z}) = H^1(G_\nu, \mathbb{Z}) = 0$ also by exercise 24. Now we turn to H^2. All we know is $0 \to H^2(X, D_X) \hookrightarrow \bigoplus H^2(\nu, \mathbb{Z})$. Now we know $H^2(\nu, \mathbb{Z}) = H^2(G_\nu, \mathbb{Z})$, and we use $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$.

Exercise 25: $H^q(G_\nu, \mathbb{Q}) = 0 \ \forall q > 0$ (as $\mathbb{Q}$ is uniquely divisible).

Hence $H^2(\nu, \mathbb{Z}) \cong H^1(\nu, \mathbb{Q}/\mathbb{Z}) = \operatorname{Hom}_{cont}(G_\nu, \mathbb{Q}/\mathbb{Z})$. This is about as much as we can say.

Step 2: Compute $H^r(X, g_* \mathbb{G}_{m,\eta})$. We again use the Leray S.S., which gives

$$(**) \quad 0 \longrightarrow H^1(X, g_* \mathbb{G}_{m,\eta}) \to \underset{\substack{\| \\ H^1(\eta, K^\times) \\ \| \\ 0 \text{ by Hilbert \#90.}}}{H^1(\eta, \mathbb{G}_{m,\eta})} \to H^0(X, R^1 g_* \mathbb{G}_{m,\eta}) \longrightarrow H^2(X, g_* \mathbb{G}_{m,\eta}) \to H^2(\eta, \mathbb{G}_{m,\eta})$$

We again claim $R^1 g_* \mathbb{G}_{m,\eta} = 0$. By the same method, we get the claim. Using $(**)$, we can extract what we need. The last piece is $H^2(\eta, \mathbb{G}_{m,\eta}) = H^2(G_k, K_s^\times) = Br(K)$.

Cor: $\operatorname{Pic}(X) \cong H^1(X_{et}, \mathbb{G}_{m,X})$.

If X is a smooth curve over $k = \bar{k}$, then we claim $H^r(X_{et}, \mathbb{G}_{m,X}) = 0$ if $r > 1$. Indeed we see $R^i i_{\nu *} \mathbb{Z} = 0$ for all $i > 0$, as ν is a closed point, $\nu = \operatorname{Spec} \bar{k}$, and since $R^i i_{\nu *} \mathbb{Z}$ = sheafification of $(U \xrightarrow{et} X) \mapsto H^i(\nu \times_X U, \mathbb{Z})$ but $\nu \times_X U = \coprod \operatorname{Spec} \bar{k}$, and has no higher cohomology.

We can use a similar technique to deduce $R^i g_* \mathbb{G}_m = 0 \ \forall i \geq 1$ (via Hilbert's 90th + Tsen's Theorem).

If X is a smooth curve over $\mathbb{F}_p$, then class field theory (?) $\Rightarrow H^2(X, \mathbb{G}_m) = 0$, $H^3(X, \mathbb{G}_m) = \mathbb{Q}/\mathbb{Z}$ and all higher ones vanish.

Discussion of $H^i(X, \mathbb{G}_m)$, $i = 1, 2$

Let G be any topological group, X a top. space. Then "usually" $H^1(X, G)$ classifies isomorphism classes of principal G-bundles on X, via cocycles.

Exercise 26:

i) If $\{\varphi_{ij}\}$ is a cocycle for a principal G-bundle, show $\{\varphi_{ij}\} \sim \{\varphi'_{ij}\}$ iff $\exists \ \{h_j : U_j \to G\}$ w/ $\varphi_{ij} = h_i \cdot \varphi'_{ij} \cdot h_j^{-1}$ is an equivalence relation and so we get a quotient set $\check{H}^1(\mathcal{U}, G)$.

ii) Prove $\check{H}^1(\mathcal{U}, G) \cong \{$iso. classes of prin. G-bundles, trivialized by $\mathcal{U}\}$.

iii) Define $\check{H}^1(X, G) = \varinjlim \check{H}^1(\mathcal{U}, G)$, taken over refinement, and show $\check{H}^1(X, G)$ classifies prin. hom. G-spaces on X.

Exercise 27: If $0 \to G' \to G \to G'' \to 0$ is a s.e.s. of top. groups, we get an exact sequence of pointed sets $0 \to G'(X) \to G(X) \to G''(X) \to \check{H}^1(X, G') \to \check{H}^1(X, G) \to \check{H}^1(X, G'')$, where $G(X) = \mathrm{Hom}(X, G)$

"Thm": Objects on X that locally are of the form $U_i \times S$, are classified by the set $\check{H}^1(X, \mathrm{Aut}\, S)$.

In particular $\check{H}^1((\mathrm{Spec}\, k)_{et}, Gl_n)$ classifies k-modules M with $M \otimes_k k_s$ a rank n k_s module, which of course are all vector spaces (all iso). Hence it vanishes.

Now $Br(k)$ classifies division algebras over k. If D & D' are two such algebras, $D' \otimes_k D$ is <u>not</u> a division algebra in general.

Def: A k-algebra A is a central simple algebra if

1) A is a simple ring
2) $\dim_k A < \infty$
3) $Z(A) = k$.

An example is $D \otimes_k D'$!(and all division algebras), or $M_n(D)$.

Thm: Any csa is isomorphic to $M_n(D)$ for some division algebra D.

Cor: CSA's form a commutative associative monoid under $- \otimes_k -$, with k the identity.

So we introduce an equivalence relation: $A_1 \sim A_2$ iff $A_1 = M_{n_1}(D)$ & $A_2 = M_{n_2}(D)$ for some n_1 & n_2.

Exercise 28:

1) A_1 & A_2 CSA's $\Rightarrow$ $A_1 \otimes_k A_2$ is a CSA,
2) A CSA $\Rightarrow$ A^{op} CSA,
3) $A \otimes_k A^{op} = M_n(k)$ for some n,
4) A a csa$/k$, $k \subset k'$, then $A \otimes_k k'$ is a CSA$/k'$.

Thm: The operation $- \otimes_k -$ descends to a group operation on CSA's$/\sim$, giving the Brauer group $Br(k)$, which is functorial in field extensions.

Def: If A is a CSA $/k$, then $k \subset k'$ is a splitting field of A if $[A] \in \ker(Br(k) \to Br(k'))$ (called $Br(k'/k)$) iff $A \otimes_k k' = M_n(k')$.

Every CSA has a splitting field (that is finite over k). $\dim_k A = n^2$, and any simple subobject B, every $\sigma \in \mathrm{Aut}\, B$ comes from $a^{-1} b a = \sigma(b)$ for some $a \in A$. In addition we can find $C \sim A$ with a subfield $K \subset C$, s.t. $[K : k] = \sqrt{\dim_k C}$, and K splits C, and $Z_C(K) = K$, and K/k is Galois.

Let $k \subset K \subset B$ as above, and $G = \mathrm{Gal}(K/k)$. Then $G \circlearrowright K$, and take some $\sigma \in G$. Then we know σ comes from conjugation $\sigma(y) = x_\sigma^{-1} y x_\sigma$ $\forall y \in K$. Can show that $\{x_\sigma \mid \sigma \in G\}$ is a K-basis of B. Moreover $x_{\sigma\tau}(x_\sigma x_\tau)^{-1} \in Z_B(K) = K$, call this element $f(\sigma, \tau)$, and so $f : G \times G \longrightarrow K^*$ and satisfies some 2-cocycle condition.

<u>Thm</u>:

1) $[f] \in H^2(G, K^*)$ depends only on $[B] \in Br(K/k) \subset Br(k)$.
2) Get a grp. hom $Br(K/k) \longrightarrow H^2(G, K^*)$,
3) It's an iso.

Passing to the limit over finite Galois extensions, we get $Br(k) = H^2(G_k, k_s^*) = H^2(\operatorname{Spec} k_{et}, \mathbb{G}_m)$, and hence $Br(k)$ is torsion.